— Quantum Optics 1 — (Class 3)

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REVIEW: SQUEEZED STATES IN COORDINATE REPRESENTATION.

Squeezing operator

$$\boxed{\begin{aligned} S^+(r)\,\hat{q}\,\hat{S}(r) &= \hat{q}\,e^{-r} \\ S^+(r)\,\hat{p}\,S(r) &= \hat{p}\,e^{+r} \end{aligned}} \Bigg\}\ \text{scales the operator}$$

$$\hat{q} \hat{=} \hat{x} = \left(\frac{\hbar}{2m\omega}\right)(a^+ + a) \qquad p \equiv \sqrt{\frac{m\omega\hbar}{2}}\,\frac{(a - a^+)}{i}$$

Photon number statistics: for a squeezed state $|\psi_r\rangle_{sq} = \hat{D}(\alpha)\hat{S}(r)|0\rangle$

First note: $$\langle\psi|\hat{H}|\psi\rangle = \underbrace{|\alpha|^2 + \tfrac{1}{2}}_{\hat{=}\ \bar{n} + \frac{1}{2}} + \underbrace{\sinh^2(r)}_{\text{function only of squeezed state}}$$

Photon number statistics: (squeezed vacuum)

$$p_n = \langle n|\hat{S}(r)|0\rangle$$

Solution: use position representation $\int_{-\infty}^{\infty} |x\rangle\langle x|\,dx = \mathbb{1}$ or $= \int |q\rangle\langle q|\,dq$

$$\langle n|\hat{S}(r)|0\rangle = \int_{-\infty}^{\infty} \langle n|x\rangle\,\underbrace{\langle x|}_{\mathbb{1}}\hat{S}(r)|0\rangle\,dx$$

$$= \int_{-\infty}^{\infty} \underbrace{\psi_n(x)}_{\text{Fock state coordinate representation}} \cdot \underbrace{\langle x|\hat{S}(r)|0\rangle}_{\hat{=}\ \psi_0(e^{r}\hat{x})}\,dx$$

(Ulf Leonhardt)

$\langle x|0\rangle = \psi_0(x)$ i.e. vacuum wave function.

Since squeezing is mathematically $\psi_0(x)$:

$$\Delta x^2 = \tfrac{1}{2}e^{-2r} \qquad \Delta p^2 = \tfrac{1}{2}e^{+2r} \qquad \Delta p \cdot \Delta x = \tfrac{1}{2}$$

$$\left\{\ \begin{aligned} \varphi(x) &= e^{+r/2}\,\psi_0(e^{r}\cdot x) \\ \varphi(p) &= e^{-r/2}\,\psi_0(e^{-r}p) \end{aligned}\right.$$

wavefunction that is squeezed

i.e Fourier transform of $\varphi(x)$.

Thus:

$$\boxed{\langle n|\hat{S}(r)|0\rangle = \int_{-\infty}^{\infty} \psi_n(x)\cdot e^{+r/2}\,\underbrace{\psi_0(e^{r}\cdot x)}\,dx}$$

Projection onto Fock State

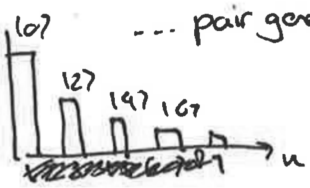
note that squeezed vacuum has $\psi_0(r) = \psi_0(-r)$

Therefore: since ψ_n has even and odd symmetry

$$P_{2m+1} = 0 \qquad m = 0, 1, 2 \ldots \Rightarrow \text{all odd terms vanish.}$$

... pair generation.

$$P_{2m} = \binom{2m}{m}\frac{1}{\cosh(r)}\cdot\left(\tfrac{1}{2}\tanh(r)\right)^{2m}$$



(i.e TWO PHOTON COHERENT STATES)

Quantum Optics I

review

squeezed states: vacuum squeezed state representation

ref. Quantum Optics an Introduction (Vogel)

repeat analysis to find Fock state $|n\rangle$ representation of the vacuum squeezed state $S(\xi)|0\rangle = |\underline{\xi},0\rangle$ squeezed state.

$$S(\xi) = e^{\frac{1}{2}(\xi^{*}\hat{a}^{2} + \xi\hat{a}^{+2})} \qquad |\xi| = r$$

(ie two photon processes). $\left|\begin{array}{l} \mu = \cosh(|r|) \\ v = e^{i\varphi}\cdot\sinh(|r|) \end{array}\right\}\ \mu^2 - |v|^2 = 1$

Without giving full operator "disentangling" calculation, this is equal to:

$$\boxed{S(r) = e^{-\frac{v}{2\mu}(a^{+})^{2}}\cdot\left(\frac{1}{\mu}\right)^{\hat{n}+\frac{1}{2}} \exp\left(\frac{v^{*}}{2\mu}\hat{a}^{2}\right)}$$

NORMAL ORDERING

Hence: $|r,0\rangle = e^{-\frac{v}{2\mu}(a^{+})^{2}}\left(\frac{1}{\mu}\right)^{\hat{n}+1/2}\cdot|0\rangle$ since $e^{\frac{v^{*}}{2\mu}\hat{a}^{2}}|0\rangle = |0\rangle$ as $a|0\rangle = 0$

$$= e^{-\frac{v}{2\mu}(a^{+})^{2}}\cdot\frac{1}{\mu^{1/2}}\left(1 + \frac{\hat{n}}{\mu} + ..\right)|0\rangle$$

$$\boxed{|r,0\rangle = \frac{1}{\sqrt{\mu}}\, e^{-v/2\mu(a^{+})^{2}}|0\rangle}$$

TWO PHOTON COHERENT STATE

ie $p_n = 0$ for $n = 2m+1$ (ie odd photon number)

Properties: revisited: (Milburn, Scully)

Operator transformations: $\boxed{S^{+}(\varepsilon) = S^{-1}(\xi) = S(-\varepsilon)}$ ie $S^{+}S = \mathbb{1}$

$$\hat{\beta} \equiv \hat{a}' = S^{+}(\xi)\,\hat{a}\,S(\xi)$$

$$= \underbrace{\hat{a}\cdot\cosh(r)}_{\mu} + \underbrace{\hat{a}^{+}\cdot\sinh(r)\cdot e^{i\varphi 2}}_{v}$$

$$\hat{\beta} \equiv \boxed{\hat{a}' = (\hat{a}\,\mu + \hat{a}^{+}v)}$$

$|\beta\rangle = D(\alpha)S(\xi)|0\rangle = |\alpha,\xi\rangle$

The squeezed states is an EIGENSTATE of the operator $\hat{\beta} \equiv (\hat{a}\mu + \hat{a}'v)$ with ~~...~~ eigenvalue.

$\alpha = \mu\beta - v\beta^{*}$

note that $[\hat{\beta},\hat{\beta}^{+}] = |\mu|^{2} - |v|^{2} = 1$ (ie commutator satisfied)

$S^{*}(\xi)|\alpha\rangle = |\alpha\rangle_{sq}$

$b|\beta_{sq}\rangle \hat{=}$

$S^{+}\hat{a}S|\beta\rangle_{sq}$

b

$= (S^{+}a)\cdot S$

Eigenvalue Equation:

$$\hat{\beta}|\beta\rangle_{sq} = \beta|\beta\rangle_{sq} \quad \text{and:} \quad \hat{\beta} = S^{+}(\xi)\,\hat{a}\,S(\xi)$$

$$\underbrace{S^{+}(\xi)\,a\,S(\xi)}_{\hat{\beta}}|\beta\rangle_{sq} = \underbrace{\beta}_{\text{eigenvalue}}|\beta\rangle_{sq} \quad \text{i.e} \Rightarrow \boxed{\underbrace{\beta}_{\text{eigenvalue}}|\beta\rangle_{sq} = S(\xi)\,\underbrace{|\beta\rangle_{\text{coherent}}}_{\text{Coherent state}}}$$

$\hat{\beta}$ (definition, derived above)

ie $a|\beta\rangle_{coh} = \beta|\beta\rangle_{coh}$.

$b = U a U^{+}$

$|\beta\rangle_{sq} = U|\beta\rangle_{coh}$.

step unclear

$|\alpha, \xi\rangle \equiv S(\xi) D(\alpha) |0\rangle$

(Mean field

$$\boxed{\begin{aligned} \mu &\equiv \cosh(r) \\ \nu &\equiv \sinh(r) e^{2i\varphi} \end{aligned}}$$

ellipse can only be rotated 180° → simils.

$$\langle a\rangle = {}_{sq}\langle \alpha, \xi | a | \alpha, \xi\rangle_{sq}$$
$$= \langle 0| D^\dagger(\alpha) S^\dagger(\xi) \hat{a} S(\xi) D(\alpha) |0\rangle$$
$$= \langle 0| D^\dagger(\alpha) (\hat{a}\mu + \hat{a}^\dagger \nu) D(\alpha) |0\rangle$$
$$= {}_{coh}\langle \alpha | \hat{a}\mu + a^\dagger \nu |\alpha\rangle_{coh} = \alpha\mu + \alpha^* \nu$$
$$\hat{=} \underline{\alpha\cosh(r) - \alpha^* e^{2i\varphi} \sinh(r)}$$

⇒ similar ~~contrast~~ to vacuum $\langle 0|\hat{a}|0\rangle = 0$ the squeezed state has also zero expectation for $\alpha = 0$ ~~but~~

BUT:

$$\langle a^\dagger a\rangle = \langle \hat{n}\rangle = {}_{sq}\langle \alpha, \xi | a^\dagger a |\alpha, \xi\rangle_{sq}$$
$$= {}_{coh}\langle \alpha | S^\dagger(\xi) a^\dagger a S(\xi) |\alpha\rangle_{coh} = \langle \alpha | S^\dagger a^\dagger \underbrace{S S^\dagger}_{\mathbb{1}} a S |\alpha\rangle \ldots$$
$$= |\alpha|^2 (\cosh^2 r + \sinh^2 r) - (\alpha^*)^2 e^{2i\varphi} \sinh(r)\cosh(r)$$
$$- e^{2i\varphi} \cosh(r)\sinh(r) + \sinh^2(r)$$

This implies that squeezed vacuum $(\alpha = 0)$

$$\boxed{\langle \hat{a}^\dagger a\rangle = \langle \xi | a^\dagger a |\xi\rangle = \sinh^2(r) = |\nu|^2}$$

Thus a SQUEEZED VACUUM has a nonvanishing photon number for $r \neq 0$ very non-intuitive. (remember $e^{\pm r}$ increase decrease noise)

~~note~~ that:

~~$\langle a\rangle = \langle \beta, \xi | \hat{a} | \xi, \beta\rangle$~~

Quadrature representation of SQUEEZED STATES

$$\boxed{\begin{aligned}\hat{X}_1 &= \tfrac{1}{2}(\hat{a}+\hat{a}^{+})\\ \hat{X}_2 &= \tfrac{1}{2i}(\hat{a}-\hat{a}^{+})\end{aligned}}$$

Note: different definitions exist $\frac{1}{2}$; $\frac{1}{\sqrt{2}}$; 1 examples.

$[\hat{X}_1, \hat{X}_2] = i/2$

or more generally

$$\boxed{\hat{X}_\varphi = \tfrac{1}{2}\left(\hat{a}e^{i\varphi} + a^{+}e^{-i\varphi}\right)}$$

$\hat{X}_{\varphi+\pi/2}$ analogous

arbitrary quadrature

with: $[\hat{X}_\varphi, \hat{X}_{\varphi+\pi/2}] = i \cdot 2$

The fluctuations of the quadrature $\hat{X}_\varphi$ are:

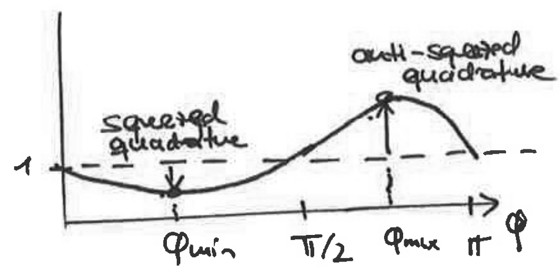
$$\langle \Delta X_\varphi^2 \rangle \equiv \langle \hat{X}_\varphi^2 \rangle - \langle \hat{X}_\varphi \rangle^2$$

and without proof: $\mu = \cosh(r) \wedge \nu = \sinh(r)\underbrace{e^{i\varphi_\varepsilon}}_{\text{squeezing phase}}$

$$_{sq}\langle \beta,\xi | \Delta\hat{X}_\varphi | \beta,\xi \rangle_{sq} = |\mu e^{i\varphi} - \nu^{*} e^{-i\varphi}|^2$$

$$= |\mu - |\nu| e^{2i\varphi+\varphi_\varepsilon}|$$

Plot:
$\varphi_\varepsilon = -\pi/2$



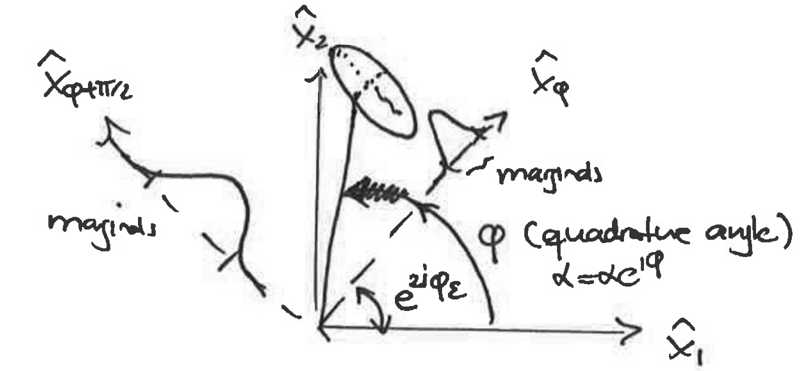
→ can be measured using HOMODYNE.

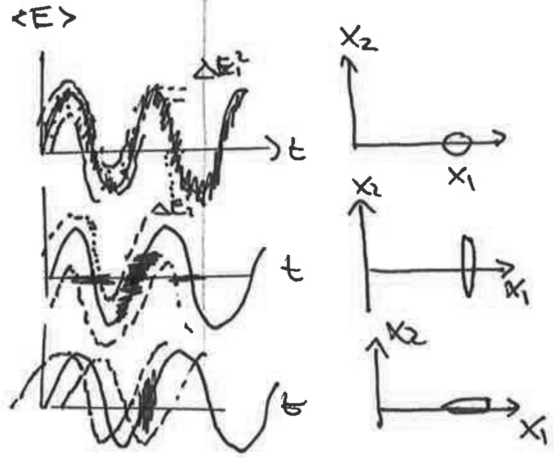
~~Experiment~~

$$\langle \beta,\xi | \Delta\hat{X}_\varphi | \beta,\xi \rangle = e^{2|r|} \quad \text{enhanced fluctuation}$$

$$\langle \beta,\xi | \Delta\hat{X}_{\varphi+\pi/2} | \beta,\xi \rangle = e^{-2|r|} \quad \text{suppressed fluctuations.}$$

Alternative graphical representation



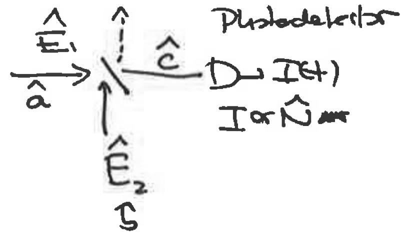


How to detect squeezed light!

~~HETERO~~ / HOMODYNE DETECTION

Single field QUADRATURE DETECTION

$$E_1(r,t) = i\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{1/2} \hat{a}\, e^{-ikr-\omega t} + \hat{a}^{+} e^{-ikr+i\omega t}$$



$$\boxed{\hat{c} = \sqrt{y}\,\hat{a} + \sqrt{1-y}\cdot\hat{b}}$$

Photodetection corresponds to $I \propto \hat{N} = \hat{c}^{+}\hat{a}$ (later in class)

$$\boxed{I \propto \langle \hat{c}^{+}\hat{c}\rangle \propto y\langle \hat{a}^{+}\hat{a}\rangle + (1-y)\langle \hat{b}^{+}\hat{b}\rangle - i\sqrt{y(1-y)}\left(\langle\hat{a}\rangle\langle\hat{b}^{+}\rangle - \langle\hat{a}^{+}\rangle\langle\hat{b}\rangle\right)}$$

independent operators → factorise

Assume 2nd E_2 field is large coherent state $|\beta e^{i\varphi}\rangle$ amplitude $|\langle\hat{b}\rangle|^2 = |\langle\beta|\hat{b}|\beta\rangle|^2 = |\beta|^2$

$$\Rightarrow \boxed{\langle\hat{c}^{+}\hat{c}\rangle = (1-y)|\beta|^2 + |\beta|\sqrt{y(1-y)}\,\langle \underbrace{\hat{a}e^{-i\varphi} + \hat{a}^{+}e^{i\varphi}}_{X_\varphi}\rangle}$$

→ one measures the Quadrature!

Subtle point: expression neglects here vacuum noise contribution ~~from~~

If there is no input $\langle a + \hat{a}^{+}\rangle = 0$ then $I \propto \langle c^{+}c\rangle = (1-y)|\beta|^2$ classical noise is in fluctuations, i.e. $\underset{I}{S(\omega)} = \int \langle I(t) I(t+T)\rangle e^{i\omega T}$ spectrum.

(~~more later~~) or $\langle\Delta\hat{u}^2\rangle$

Lecture-3-

Quantum Optics I

PHASE SPACE METHODS

Motivation for phase space Methods. In many calcs instead of computing an operator expectation value $\hat{G}(\hat{a},\hat{a}^+)$ i.e $G(\hat{x},\hat{p})$

$$\langle G(\hat{x},\hat{p}^+)\rangle = \int \psi^*(x)\, \underbrace{G(\hat{x},\hat{p}^+)}\, \psi(x)\, dx$$

via the operator formalism (ie $\hat{x}\to x$; $\vec{p}\to -i\hbar\vec{\nabla}$)

one computes the expectation via a quasiprobability function:

$$\langle \hat{G}\rangle = \iint \underbrace{g(q,p)}_{\text{Quasi-probability Fct.}}\ \underbrace{F(q,p)}_{\text{"Distribution function" of the State}}\, dq\, dp$$

Examples for normally ordered operators (that is $:\hat{a}^+\hat{a}+\hat{a}\hat{a}^+: = 2\hat{a}^+\hat{a}$, in contrast to SYMMETRIZED

$$\overline{\hat{a}^+\hat{a}} = \frac{1}{2}(\hat{a}^+\hat{a}+\hat{a}\hat{a}^+)$$

occur in photodetection e.g Hanbury-Twiss-Townes:

$$\langle\psi|\, a^+(t)\, a^+(t+\tau)\, a(t+\tau)\, a(t)\,\rangle$$

is the Intensity autocorrelation function. e.g $\langle:\hat{I}(t+\tau)I(t):\rangle$

Hence photon correlations involve NORMALLY ORDERED operators.

(Stoch. Methods Gardiner Ch. X).

Operator expansions in coherent states eg calculation of $\boxed{\hat{T} = (a^+)^m(\hat{a})^n}$ expectation values for coherent states.

~~$\hat{T} \leq \mathbb{1}\hat{T}\mathbb{1} = \frac{1}{\pi^2}\iint d^2\alpha\, d^2\beta\, |\alpha\rangle\langle\alpha|\hat{T}|\beta\rangle\langle\beta|$~~

$\langle\hat{T}\rangle$ DENSITY MATRIX AND PROBABILITIES

For a pure state $\langle\hat{M}\rangle = \langle\psi|\hat{M}|\psi\rangle$ $\qquad \rho \equiv \sum_i p(i)|\psi_i\rangle\langle\psi_i|$

for a non pure state $\langle\hat{M}\rangle = \sum_i p(i)\langle\psi_i|\hat{M}|\psi_i\rangle \hat{=} Tr(\rho M)$

Then: $Tr\{\rho M\} = \sum_n\sum_i p(i)\langle n|\psi_i\rangle\langle\psi_i|\hat{M}|n\rangle$ $\qquad \mathbb{1} = \sum |n\rangle\langle n|$

$$= \sum_n\sum_i p(i)\langle\psi_i|\hat{M}\underbrace{|n\rangle\langle n|\psi_i\rangle}_{\psi_i} \hat{=} \sum p(i)\langle\psi_a|M\psi_a\rangle \equiv \langle M\rangle$$

Quantum Optics

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Properties: $Tr\{\rho H\} = Tr\{H\rho\}$ cyclic

(i) $Tr\{\rho\} = 1$ for $\sum_i p_i \langle\psi_i|\psi_i\rangle = \sum p_i = 1$

(ii) Pure state $\rho^2 = \rho$

Time evolution:

$$H|\psi\rangle = -i\hbar\partial_t|\psi\rangle$$

$$\Rightarrow \partial_t\rho = \sum_i p_i\left(\underbrace{\partial_t|\psi_i\rangle}_{i\hbar\partial_t|\psi_i\rangle = \hat{H}|\psi_i\rangle}\langle\psi_i| + |\psi_i\rangle\underbrace{\langle\partial_t\psi_i|}_{-i\hbar\partial_t\langle\psi| = \langle\psi|\hat{H}}\right)$$

$\Rightarrow$ $\boxed{i\hbar\partial_t\rho = (\hat{H}\rho - \rho\hat{H}) = [\hat{H},\rho]}$ von NEUMANN EQUATION

and $\boxed{\rho(t) = e^{-i\hat{H}t/\hbar}\rho(0)e^{+i\hat{H}/\hbar \cdot t}}$

PHASE SPACE Representations: R-representation

Operator $\langle\hat{O}\rangle = \langle f(\hat{a},\hat{a}^\dagger)\rangle = Tr\{\rho\hat{f}(\hat{a},\hat{a}^\dagger)\}$

The density matrix is (in Fock states)

$$\rho = \sum_{n,m}\rho_{nm}|n\rangle\langle m| = \sum\langle n|\rho|m\rangle |n\rangle\langle m|$$

alternatively (glauber) $\mathbb{1} = \frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\,\partial^2\alpha$ thus:

$$\rho = \frac{1}{\pi^2}\iint\partial^2\alpha\,\partial^2\beta\,\rho(\alpha,\beta)\,|\alpha\rangle\langle\beta|$$

and $\langle\alpha|\hat{\rho}|\beta\rangle = \rho(\alpha,\beta)$ and $a\to\alpha \wedge a^\dagger\to\alpha^*$ in function f.

we thus have:

$$\langle\hat{O}\rangle = \frac{1}{\pi^2}\iint\partial^2\alpha\,\partial^2\beta\,\underbrace{\rho(\alpha,\beta)}_{\text{density matrix}}\underbrace{\langle\beta|f(a,a^\dagger)|\alpha\rangle}_{f(\alpha,\alpha^*)}$$

$\rightarrow$ normally ordered follows immediately. $a^\dagger a \hat{=} \beta^*\alpha$

$\Rightarrow$ quasi-probability distribution.

i.e. ~~[illegible]~~

Glauber called this the R-representation $\boxed{R(\alpha,\beta^*) \equiv \langle\alpha|\rho|\beta\rangle e^{1/2(|\alpha|^2+|\beta|^2)}}$ $\hat{=}\sum_{n,m}\frac{\langle n|\rho|m\rangle\alpha^{*n}\beta^{*m}}{\sqrt{n!m!}}$ in Fock state Basis

Thus: $\boxed{\rho = \iint\frac{1}{\pi^2}\partial\alpha^2\partial\beta^2\,|\alpha\rangle\langle\beta|\,R(\alpha^*,\beta)\,e^{-1/2|\alpha|^2+|\beta|^2}}$

and $\langle n|\rho|m\rangle = \frac{1}{\pi^2}\int R(\alpha^*,\beta)\sqrt{n!m!}^{(-1)}\alpha^n\beta^{*m}e^{-|\alpha|^2-|\beta|^2}\partial^2\alpha\,\partial^2\beta$ Fock State Basis.